

LINEARIZED GRAVITY AS A GAUGE THEORY

J. A. Nieto¹

*Facultad de Ciencias Físico-Matemáticas de la Universidad Autónoma
de Sinaloa, 80010 Culiacán Sinaloa, México*

Abstract

We discuss linearized gravity from the point of view of a gauge theory. In (3+1)-dimensions our analysis allows to consider linearized gravity in the context of the MacDowell-Mansouri formalism. Our observations may be of particular interest in the strong-weak coupling duality for linearized gravity, in Randall-Sundrum brane world scenario and in Ashtekar formalism.

Keywords. linearized gravity; gauge theory

Pacs numbers: 04.20.-q, 04.60.+n, 11.15.-q, 11.10.Kk

November, 2003

¹nieto@uas.uasnet.mx

1.- INTRODUCTION

Among the physical features of linearized gravity its similarity with the source free Maxwell theory seems to be one of the most interesting. As an example of this fact, recently it was shown [1] that the strong-weak duality for linearized gravity makes sense as in the Maxwell theory. In this scenario the cosmological constant, or the Planck length, plays the role of the charge duality transformation. At this respect, it is worth mentioning that this strong-weak duality, or S-duality, for linearized gravity seems to have inspired other related works [2] and [3-4].

Maxwell theory is a $U(1)$ gauge theory and therefore one should expect to be able to identify linearized gravity with some consistent gauge theory. For instance, in a linearized version of Ashtekar formulation [5], linearized gravity can be identified with $U(1) \times U(1) \times U(1)$ abelian gauge theory. However, this is a canonical construction and it is more appropriate to compare it with the corresponding canonical Maxwell theory. In this work we investigate an alternative gauge theory for linearized gravity. Our approach has the advantage that can be applied in the context of MacDowell-Mansouri formalism [6]. The present investigation may be of special interest in Randall-Sundrum brane world scenario [7], in gravitational waves formalism (see [8] and references there in) and in quantum linearized gravity [9]. This last possibility is of particular interest for the understanding of different aspects of quantum gravity as has been shown by Hartle [10]. Since the quantum aspects of an abelian theory are better understood it appears interesting to have a gauge theory interpretation of linearized gravity.

Another source of interest in linearized gravity as a gauge theory comes from the observation [11] that linearized gravity in four dimensions can be understood as an irreducible representation of the de Sitter group $S(4,1)$. We shall show that since the cosmological constant enters in a natural way in MacDowell-Mansouri theory, our formalism leads to the de Sitter space, and therefore to the de Sitter group symmetry $S(4,1)$. Moreover, since recently, Randall-Sundrum brane world scenario [12] has motivated [13]-[16] the study of a linearized gravity in different backgrounds and in particular in de Sitter (or anti de Sitter) background, the present work may be interesting in this direction.

Our work may be also useful to clarify some aspects about the relation between the mass of the graviton and the cosmological constant which recently has been subject of some controversy [17]-[18] in connection with causality for the propagation of a graviton in an electromagnetic background. This is because, once again, the cosmological constant, in the linearized MacDowell-Mansouri theory, arises in a natural way as we shall show in this work.

The plan of this work is as follows: In section 2, we discuss the traditional way to

see linearized gravity as an abelian gauge theory. In sections 3 and 4, we propose an alternative theory of linearized gravity as gauge theory. In section 5, we apply our results to the case of linearized MacDowell-Mansouri theory. Finally, in section 6, we make some final comments.

2.- LINEARIZED GRAVITY AS A GAUGE THEORY

Here, we closely follow reference [1]. In terms of the linearized metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (1)$$

where $h_{\mu\nu}$ is a small deviation of the metric $g_{\mu\nu}$ from the Minkowski metric

$$(\eta_{\mu\nu}) = \text{diag}(-1, 1, \dots, 1),$$

the first-order curvature tensor becomes

$$F_{\mu\nu\alpha\beta} = -\frac{1}{2}(\partial_\mu\partial_\alpha h_{\nu\beta} - \partial_\mu\partial_\beta h_{\nu\alpha} - \partial_\nu\partial_\alpha h_{\mu\beta} + \partial_\nu\partial_\beta h_{\mu\alpha}), \quad (2)$$

which is invariant under the transformation

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \quad (3)$$

Here, ξ_μ is an arbitrary vector field and $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$.

It is not difficult to see that $F_{\mu\nu\alpha\beta}$ satisfies the following relations:

$$F_{\mu\nu\alpha\beta} = -F_{\mu\nu\beta\alpha} = -F_{\nu\mu\alpha\beta} = F_{\alpha\beta\mu\nu},$$

$$F_{\mu\nu\alpha\beta} + F_{\mu\beta\nu\alpha} + F_{\mu\alpha\beta\nu} = 0, \quad (4)$$

$$\partial_\lambda F_{\mu\nu\alpha\beta} + \partial_\mu F_{\nu\lambda\alpha\beta} + \partial_\nu F_{\lambda\mu\alpha\beta} = 0.$$

It is interesting to observe that, in 3+1 dimensions, the dual $*F_{\mu\nu\alpha\beta} \equiv \frac{1}{2}\epsilon_{\mu\nu\tau\sigma}F_{\alpha\beta}^{\tau\sigma}$, where $\epsilon_{\mu\nu\alpha\beta}$ is a completely antisymmetric Levi-Civita tensor with $\epsilon_{0123} = -1$ and the indices are raised and lowered by means of $\eta^{\alpha\beta}$ and $\eta_{\alpha\beta}$, does not satisfy the relations (4) unless $F_{\mu\alpha\nu}^\alpha$ satisfies the vacuum Einstein equations $F_{\mu\nu} = F_{\mu\alpha\nu}^\alpha = 0$.

Let us introduce the ‘gauge’ field

$$A_{\mu\alpha\beta} = \frac{1}{2}(\partial_\beta h_{\mu\alpha} - \partial_\alpha h_{\mu\beta}). \quad (5)$$

Observe that $A_{\mu\alpha\beta} = -A_{\mu\beta\alpha}$. Using (3), we find that $A_{\mu\alpha\beta}$ transforms as

$$\delta A_{\mu\alpha\beta} = \partial_\mu \lambda_{\alpha\beta}, \quad (6)$$

where $\lambda_{\alpha\beta} = \frac{1}{2}(\partial_\beta \xi_\alpha - \partial_\alpha \xi_\beta) = -\lambda_{\beta\alpha}$.

The expression (5) allows to write the curvature tensor $F_{\mu\nu}^{\alpha\beta}$ as

$$F_{\mu\nu}^{\alpha\beta} = \partial_\mu A_\nu^{\alpha\beta} - \partial_\nu A_\mu^{\alpha\beta}. \quad (7)$$

Thus, we have shown that the tensor $F_{\mu\nu}^{\alpha\beta}$ may be written in the typical form of an abelian Maxwell field strength $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a$, where the index a runs over some abelian group such as $U(1) \times U(1) \times \dots \times U(1)$. Note that $F_{\mu\nu}^{\alpha\beta}$ is invariant under the transformation $\delta A_\mu^{\alpha\beta} = \partial_\mu \lambda^{\alpha\beta}$ which has exactly the same form as the transformation of abelian gauge fields $\delta A_\mu^a = \partial_\mu \lambda^a$, where λ^a is an arbitrary function of the coordinates x^μ .

3.- ALTERNATIVE DESCRIPTION OF LINEARIZED GRAVITY AS A GAUGE THEORY

The motivation for the present work arose from the observation that although $F_{\mu\nu}^{\alpha\beta}$ has the typical form of an abelian field strength, $A_{\mu\alpha\beta}$ has also such a form. In fact, some authors (see [17] and references therein) take the combination $A_{\mu\alpha\beta} A^{\mu\alpha\beta}$ as part of the kinetic term in the Fierz-Pauli action [19]. In what follows we shall show that in fact $F_{\mu\nu}^{\alpha\beta}$ and $A_{\mu\alpha\beta}$ can be associated to the curvature of a gauge field, but while $F_{\mu\nu}^{\alpha\beta}$ can be identified with the genuine reduced curvature, $A_{\mu\alpha\beta}$ is related to the torsion. We shall also clarify the group aspects associated to $F_{\mu\nu}^{\alpha\beta}$ and $A_{\mu\alpha\beta}$.

Consider a $SO(n, 1)$ gravitational gauge field $\omega_\mu^{AB} = -\omega_\mu^{BA}$, where the index μ runs from 0 to $n-1$. Assume that such a gauge field is broken into a $SO(n-1, 1)$ gauge field connection ω_μ^{ab} and the $\omega_\mu^{na} = e_\mu^a$ vierbein field, with n as a fixed index. With these notation we find that the $SO(n, 1)$ curvature

$$\mathcal{R}_{\mu\nu}^{AB} = \partial_\mu \omega_\nu^{AB} - \partial_\nu \omega_\mu^{AB} + \omega_\mu^{AC} \omega_{\nu C}^B - \omega_\nu^{AC} \omega_{\mu C}^B \quad (8)$$

leads to

$$\mathcal{R}_{\mu\nu}^{ab} = R_{\mu\nu}^{ab} - \Sigma_{\mu\nu}^{ab} \quad (9)$$

and

$$\mathcal{R}_{\mu\nu}^{na} = \partial_\mu e_\nu^a - \partial_\nu e_\mu^a + \omega_\mu^{ac} e_{\nu c} - \omega_\nu^{ac} e_{\mu c}, \quad (10)$$

where

$$R_{\mu\nu}^{ab} = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + \omega_\mu^{ac} \omega_{\nu c}^b - \omega_\nu^{ac} \omega_{\mu c}^b \quad (11)$$

is the $SO(n-1, 1)$ curvature and

$$\Sigma_{\mu\nu}^{ab} = e_\mu^a e_\nu^b - e_\nu^a e_\mu^b. \quad (12)$$

It turns out that $T_{\mu\nu}^a \equiv \mathcal{R}_{\mu\nu}^{na}$ can be identified with the torsion.

Let us now assume a vanishing torsion; $T_{\mu\nu}^a = 0$. From (10) it follows that

$$\partial_\mu e_\nu^a - \Gamma_{\mu\nu}^\alpha e_\alpha^a + \omega_\mu^{ac} e_{\nu c} = 0, \quad (13)$$

where $\Gamma_{\mu\nu}^\alpha = \Gamma_{\nu\mu}^\alpha$. Using (13) we find that $\omega_{\mu\nu\alpha} = e_{\nu a} e_{\alpha b} \omega_\mu^{ab}$ becomes

$$\omega_{\mu\nu\alpha} = \frac{1}{2}(e_{\alpha a} \Lambda_{\mu\nu}^a + e_{\mu a} \Lambda_{\alpha\nu}^a + e_{\nu a} \Lambda_{\alpha\mu}^a), \quad (14)$$

where

$$\Lambda_{\mu\nu}^a = \partial_\mu e_\nu^a - \partial_\nu e_\mu^a. \quad (15)$$

The vierbein field e_μ^a and the metric $g_{\mu\nu}$ can be related through the formula

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}, \quad (16)$$

where η_{ab} is an internal Minkowski metric.

So far we have not considered any approximation. In the linear case, we can write $e_\mu^a = b_\mu^a + l_\mu^a$. Thus, considering that $\eta_{\mu\nu} = b_\mu^a b_\nu^b \eta_{ab}$, at first order the expression (16) becomes

$$g_{\mu\nu} = \eta_{\mu\nu} + b_\mu^a l_\nu^b \eta_{ab} + b_\nu^a l_\mu^b \eta_{ab}. \quad (17)$$

Therefore, by comparing this expression with (1) we find

$$h_{\mu\nu} = l_{\mu\nu} + l_{\nu\mu}. \quad (18)$$

Here, we used the definition $l_{\mu\nu} \equiv b_\nu^a l_\mu^b \eta_{ab}$. To first order the formula (14) becomes

$$\omega_{\mu\nu\alpha} = \frac{1}{2}(b_{\alpha a}(\partial_\mu l_\nu^a - \partial_\nu l_\mu^a) + b_{\mu a}(\partial_\alpha l_\nu^a - \partial_\nu l_\alpha^a) + b_{\nu a}(\partial_\alpha l_\mu^a - \partial_\mu l_\alpha^a)). \quad (19)$$

Considering that $\partial_\mu b_{\alpha a} = 0$ and that $l_{\mu\nu} = b_{\nu a} l_\mu^a$ we find that this expression can also be written as

$$\omega_{\mu\nu\alpha} = \frac{1}{2}((\partial_\mu l_{\nu\alpha} - \partial_\nu l_{\mu\alpha}) + (\partial_\alpha l_{\nu\mu} - \partial_\nu l_{\alpha\mu}) + (\partial_\alpha l_{\mu\nu} - \partial_\mu l_{\alpha\nu})). \quad (20)$$

Now using (5) and (18) we discover that (20) can be written as

$$\omega_{\mu\nu\alpha} = A_{\mu\nu\alpha} + \frac{1}{2}\partial_\mu f_{\nu\alpha}, \quad (21)$$

where

$$f_{\nu\alpha} = (l_{\nu\alpha} - l_{\alpha\nu}). \quad (22)$$

Therefore, we have shown that up to a gauge transformation $\omega_{\mu\nu\alpha}$ and $A_{\mu\nu\alpha}$ describe the same physics. From (7), we also find that to first order

$$R_{\mu\nu}^{ab} = b_\alpha^a b_\beta^b F_{\mu\nu}^{\alpha\beta}. \quad (23)$$

With these results at hand we observe that the reason that $A_{\mu\nu\alpha}$ has the form given in (5) is because the torsion vanishes. Now that we have the links $\omega_{\mu\nu\alpha} \leftrightarrow A_{\mu\nu\alpha}$ and $R_{\mu\nu}^{ab} \leftrightarrow F_{\mu\nu}^{\alpha\beta}$ we may proceed to analyze the group structure underling such links.

4.- GROUP STRUCTURE IN LINEARIZED GRAVITY

In order to capture the group information in the development of the previous section we introduce the generator $J_{AB} = -J_{BA}$ associated to the group $SO(n, 1)$ and we write ω_μ and $\mathcal{R}_{\mu\nu}$ as

$$\omega_\mu = \frac{1}{2} \omega_\mu^{AB} J_{AB} \quad (24)$$

and

$$\mathcal{R}_{\mu\nu} = \frac{1}{2} \mathcal{R}_{\mu\nu}^{AB} J_{AB}, \quad (25)$$

respectively.

It turns out that $\mathcal{R}_{\mu\nu}$ can be written in terms of ω_μ in the form

$$\mathcal{R}_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + \frac{1}{4} [\omega_\mu, \omega_\nu], \quad (26)$$

where $[\omega_\mu, \omega_\nu]$ is determined by the Lie algebra

$$[J_{AB}, J_{CD}] = \frac{1}{2} C_{ABCD}^{EF} J_{EF}. \quad (27)$$

Here, C_{ABCD}^{EF} are the structure constants associated to $SO(n, 1)$. This algebra can be written as

$$[J_{ab}, J_{cd}] = \frac{1}{2} C_{abcd}^{ef} J_{ef}, \quad (28)$$

$$[J_{na}, J_{cd}] = C_{nacd}^{nf} J_{nf} = \delta_{ac} J_{nd} - \delta_{ad} J_{nc}, \quad (29)$$

$$[J_{na}, J_{nb}] = \frac{1}{2} C_{nabn}^{ef} J_{ef} = J_{ab}. \quad (30)$$

Using (28)-(30) we find that (26) becomes

$$\begin{aligned}
\mathcal{R}_{\mu\nu} = & \frac{1}{2}(\partial_\mu\omega_\nu^{ab} - \partial_\nu\omega_\mu^{ab})J_{ab} + (\partial_\mu\omega_\nu^{nb} - \partial_\nu\omega_\mu^{nb})J_{nb} \\
& + \frac{1}{4}[J_{ab}, J_{cd}]\omega_\mu^{ab}\omega_\nu^{cd} + \frac{1}{2}[J_{na}, J_{cd}](\omega_\mu^{na}\omega_\nu^{cd} - \omega_\mu^{cd}\omega_\nu^{na}) \\
& + [J_{na}, J_{nb}]\omega_\mu^{na}\omega_\nu^{nb}.
\end{aligned} \tag{31}$$

In order to study the linearized approximation of (31) let us write $\omega_\mu^{ab} \rightarrow \frac{1}{\lambda}\omega_\mu^{ab}$ and $\omega_\mu^{na} \rightarrow b_\mu^a + \frac{1}{\lambda}l_\mu^a$ where λ is an auxiliary parameter measuring the order of approximation. The expression (31) becomes

$$\begin{aligned}
\mathcal{R}_{\mu\nu} = & \frac{1}{2\lambda}(\partial_\mu\omega_\nu^{ab} - \partial_\nu\omega_\mu^{ab})J_{ab} + \frac{1}{\lambda}(\partial_\mu l_\nu^a - \partial_\nu l_\mu^a)J_{na} \\
& + \frac{1}{4\lambda^2}[J_{ab}, J_{cd}]\omega_\mu^{ab}\omega_\nu^{cd} + \frac{1}{2\lambda}[J_{na}, J_{cd}](b_\mu^a\omega_\nu^{cd} - \omega_\mu^{cd}b_\nu^a) + \frac{1}{2\lambda^2}[J_{na}, J_{cd}](l_\mu^a\omega_\nu^{cd} - \omega_\mu^{cd}l_\nu^a) \\
& + [J_{na}, J_{nb}]b_\mu^a b_\nu^b + \frac{1}{\lambda}[J_{na}, J_{nb}](b_\mu^a l_\nu^b - l_\mu^b b_\nu^a) + \frac{1}{\lambda^2}[J_{na}, J_{nb}]l_\mu^a l_\nu^b.
\end{aligned} \tag{32}$$

From this expression it is evident that when $\lambda \rightarrow \infty$ the term $\frac{1}{4\lambda^2}[J_{ab}, J_{cd}]\omega_\mu^{ab}\omega_\nu^{cd}$ may be interpreted as a Wigner contraction leading to $[J_{ab}, J_{cd}] = 0$ in the limit $\lambda \rightarrow \infty$ and therefore the group structure with respect to this bracket in the limit $\lambda \rightarrow \infty$ corresponds to an abelian theory. However, we observe that since there are both factors $\frac{1}{\lambda}$ and $\frac{1}{\lambda^2}$ in connection with the brackets $[J_{na}, J_{cd}]$ and $[J_{na}, J_{nb}]$, it is not possible to apply a Wigner procedure to these two brackets and therefore the theory do not admit the interpretation of an abelian theory with respect these two brackets. From these observations it follows that in the limit $\lambda \rightarrow \infty$ the algebra (28)-(30) can be taken as

$$[J_{ab}, J_{cd}] = 0, \tag{33}$$

$$[J_{na}, J_{cd}] = C_{nacd}^{nf}J_{nf} = \delta_{ac}J_{nd} - \delta_{ad}J_{nc}, \tag{34}$$

$$[J_{na}, J_{nb}] = \frac{1}{2}C_{nabf}^{ef}J_{ef} = J_{ab}. \tag{35}$$

Thus, we have shown that it is consistent to substitute from the beginning the algebra (33)-(35) into (32) and then to proceed to eliminate terms of order $\frac{1}{\lambda^2}$. In a essence what we have proved is that in the limit $\lambda \rightarrow \infty$ it makes sense to take the subgroup $SO(n-1, 1)$ of $SO(n, 1)$ as a ‘collapsed’ abelian group.

5.- THE ACTION STRUCTURE FOR A GAUGE THEORY OF LINEARIZED GRAVITY IN FOUR DIMENSIONS

So far we have analyzed linearized gravity at the level of gauge field and its correspondent curvature. In order to complete our analysis we need to introduce an action. In this scenario the Lovelock theory [20] seems to be the most indicated for our goal. Just to avoid unnecessary complications in the general case, here, we focus on the Lovelock theory in four dimensions which has become known as a MacDowell-Mansouri theory [6].

In four dimensions the gauge group $SO(n, 1)$ becomes the de Sitter group $SO(4, 1)$. In this case, the gravitational gauge field ω_μ^{AB} is broken into the $SO(3, 1)$ connection ω_μ^{ab} and the $\omega_\mu^{4a} = e_\mu^a$ tetrad field. Thus, in the weak field approximation, the $SO(4, 1)$ de Sitter curvature (8) leads to

$$\mathcal{R}_{\mu\nu}^{ab} = R_{\mu\nu}^{ab} - \Sigma_{\mu\nu}^{ab} \quad (36)$$

and

$$\mathcal{R}_{\mu\nu}^{4a} = \partial_\mu l_\nu^a - \partial_\nu l_\mu^a + \omega_\mu^{ac} b_{\nu c} - \omega_\nu^{ac} b_{\mu c}, \quad (37)$$

where

$$R_{\mu\nu}^{ab} = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} \quad (38)$$

is the linearized curvature and

$$\Sigma_{\mu\nu}^{ab} = b_\mu^a b_\nu^b - b_\nu^a b_\mu^b + b_\mu^a l_\nu^b - b_\nu^a l_\mu^b + l_\mu^a b_\nu^b - l_\nu^a b_\mu^b. \quad (39)$$

Here, in the computation of $\Sigma_{\mu\nu}^{ab}$, we have used the relation $e_\mu^a = b_\mu^a + l_\mu^a$ and the approximation $l_\mu^a l_\nu^b = l_\nu^b l_\mu^a = 0$. From (39) we find that $\Sigma_{\mu\nu}^{\alpha\beta} = b_\mu^\alpha b_\nu^\beta \Sigma_{\mu\nu}^{ab}$ is given by

$$\Sigma_{\mu\nu}^{\alpha\beta} = \delta_{\mu\nu}^{\alpha\beta} + \Omega_{\mu\nu}^{\alpha\beta}, \quad (40)$$

where

$$\delta_{\mu\nu}^{\alpha\beta} = \delta_\mu^\alpha \delta_\nu^\beta - \delta_\mu^\beta \delta_\nu^\alpha \quad (41)$$

and

$$\Omega_{\mu\nu}^{\alpha\beta} = \delta_\mu^\alpha l_\nu^\beta - \delta_\nu^\alpha l_\mu^\beta + l_\mu^\alpha \delta_\nu^\beta - l_\nu^\alpha \delta_\mu^\beta. \quad (42)$$

The MacDowell-Mansouri's action is

$$S = \frac{1}{4} \int d^4x \varepsilon^{\mu\nu\alpha\beta} \mathcal{R}_{\mu\nu}^{ab} \mathcal{R}_{\alpha\beta}^{cd} \epsilon_{abcd}, \quad (43)$$

where $\varepsilon^{\mu\nu\alpha\beta}$ is the completely antisymmetric tensor associated to the space-time, with $\varepsilon^{0123} = 1$ and $\varepsilon_{0123} = 1$, while ϵ_{abcd} is also the completely antisymmetric tensor but now associated to the internal group $S(3, 1)$, with $\epsilon_{0123} = -1$. We assume that the

internal metric is given by $(\eta_{ab}) = (-1, 1, 1, 1)$. Therefore, we have $\epsilon^{0123} = 1$. It is well known that, in the general case, the action (43) leads to three terms; the Hilbert Einstein action, the cosmological constant term and the Euler topological invariant (or Gauss-Bonnet term). It is worth mentioning that the action (43) may also be obtained considering Lovelock theory [20] in four dimensions.

In the linearized case the action (43) can be written as

$$S = \frac{1}{4} \int d^4x \epsilon^{\mu\nu\alpha\beta} \mathcal{R}_{\mu\nu}^{\alpha\beta} \mathcal{R}_{\alpha\beta}^{\rho\sigma} \epsilon_{\alpha\beta\rho\sigma}, \quad (44)$$

with $\mathcal{R}_{\mu\nu}^{\alpha\beta} = b_a^\alpha b_b^\beta \mathcal{R}_{\mu\nu}^{ab}$. Using (36) and (40) we find

$$\begin{aligned} S = & \frac{1}{4} \int d^4x \epsilon^{\mu\nu\alpha\beta} R_{\mu\nu}^{\gamma\delta} R_{\alpha\beta}^{\rho\sigma} \epsilon_{\gamma\delta\rho\sigma} - \frac{1}{2} \int d^4x \epsilon^{\mu\nu\alpha\beta} \delta_{\mu\nu}^{\gamma\delta} R_{\alpha\beta}^{\rho\sigma} \epsilon_{\gamma\delta\rho\sigma} \\ & - \frac{1}{2} \int d^4x \epsilon^{\mu\nu\alpha\beta} \Omega_{\mu\nu}^{\gamma\delta} R_{\alpha\beta}^{\rho\sigma} \epsilon_{\gamma\delta\rho\sigma} + \frac{1}{4} \int d^4x \epsilon^{\mu\nu\alpha\beta} \delta_{\mu\nu}^{\gamma\delta} \delta_{\alpha\beta}^{\rho\sigma} \epsilon_{\gamma\delta\rho\sigma} \\ & + \frac{1}{2} \int d^4x \epsilon^{\mu\nu\alpha\beta} \Omega_{\mu\nu}^{\gamma\delta} \delta_{\alpha\beta}^{\rho\sigma} \epsilon_{\gamma\delta\rho\sigma} + \frac{1}{4} \int d^4x \epsilon^{\mu\nu\alpha\beta} \Omega_{\mu\nu}^{\gamma\delta} \Omega_{\alpha\beta}^{\rho\sigma} \epsilon_{\gamma\delta\rho\sigma}. \end{aligned} \quad (45)$$

From (45) and (38) we observe that the second terms is total derivatives. The fourth term is just a constant and therefore can be dropped from the action. The fifth term is proportional to $tr(l_\beta^\alpha) = \frac{1}{2}tr(h_\beta^\alpha)$ and can be dropped in a gauge fixing $tr(h_\beta^\alpha) = 0$. The first term is also a total derivative but it is useful to keep it for S-duality considerations [1]. Therefore, considering the dynamic and topological important terms the action (45) is reduced to

$$\begin{aligned} S = & \frac{1}{4} \int d^4x \epsilon^{\mu\nu\alpha\beta} R_{\mu\nu}^{\gamma\delta} R_{\alpha\beta}^{\rho\sigma} \epsilon_{\gamma\delta\rho\sigma} - \frac{1}{2} \int d^4x \epsilon^{\mu\nu\alpha\beta} \Omega_{\mu\nu}^{\gamma\delta} R_{\alpha\beta}^{\rho\sigma} \epsilon_{\gamma\delta\rho\sigma} \\ & + \frac{1}{4} \int d^4x \epsilon^{\mu\nu\alpha\beta} \Omega_{\mu\nu}^{\gamma\delta} \Omega_{\alpha\beta}^{\rho\sigma} \epsilon_{\gamma\delta\rho\sigma}. \end{aligned} \quad (46)$$

This action was the starting point in reference [1] in connection with the S-duality for linearized gravity. Using (41) and (42) we find

$$\begin{aligned} S = & \frac{1}{4} \int d^4x \epsilon^{\mu\nu\alpha\beta} R_{\mu\nu}^{\gamma\delta} R_{\alpha\beta}^{\rho\sigma} \epsilon_{\gamma\delta\rho\sigma} - 2 \int d^4x \epsilon^{\mu\nu\alpha\beta} \delta_\mu^\tau l_\nu^\lambda R_{\alpha\beta}^{\sigma\rho} \epsilon_{\tau\lambda\sigma\rho} \\ & + 4 \int d^4x \epsilon^{\mu\nu\alpha\beta} \delta_\mu^\tau l_\nu^\lambda \delta_\alpha^\sigma l_\beta^\rho \epsilon_{\tau\lambda\sigma\rho}. \end{aligned} \quad (47)$$

Since $\epsilon^{\mu\nu\alpha\beta} \delta_\mu^\tau \epsilon_{\tau\lambda\sigma\rho} = -\delta_{\lambda\sigma\rho}^{\nu\alpha\beta}$ and $\epsilon^{\mu\nu\alpha\beta} \delta_\mu^\tau \delta_\alpha^\sigma \epsilon_{\tau\lambda\sigma\rho} = -2\delta_{\lambda\rho}^{\nu\beta}$, where in general $\delta_{\tau\lambda\sigma\rho}^{\mu\nu\alpha\beta}$ is a generalized delta, we discover that (47) can be written as

$$\begin{aligned} S = & \frac{1}{4} \int d^4x \epsilon^{\mu\nu\alpha\beta} R_{\mu\nu}^{\tau\lambda} R_{\alpha\beta}^{\sigma\rho} \epsilon_{\tau\lambda\sigma\rho} - 4 \int d^4x h^{\mu\nu} (R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R) \\ & - 2 \int d^4x (h^2 - h^{\mu\nu} h_{\mu\nu}) - 2 \int d^4x f^{\mu\nu} f_{\mu\nu}. \end{aligned} \quad (48)$$

Here, we used the following definitions: $R_{\mu\nu} \equiv \eta^{\alpha\beta} R_{\mu\alpha\nu\beta}$, $R \equiv \eta^{\mu\nu} \eta^{\alpha\beta} R_{\mu\alpha\nu\beta}$ and $h \equiv \eta^{\mu\nu} h_{\mu\nu} = 2\eta^{\mu\nu} l_{\mu\nu}$. Further, we considered that $R_{\mu\nu} = R_{\nu\mu}$ and $h^{\mu\nu} R_{\mu\nu} = 2l^{\mu\nu} R_{\mu\nu}$. We recognize the second term and the third term in (48) as the Einstein action for linearized gravity with cosmological constant, while the first term is a total derivative. The last term in (48) becomes a total derivative under the transformation $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$, given in (3), since in this case according to (6) and (21) one may have $f_{\mu\nu} = (\partial_\mu \xi_\nu - \partial_\nu \xi_\mu)$.

It is worth mentioning that in order to study S-duality for linearized gravity the action (44) was generalized in the form [1]

$$\mathcal{S} = \frac{1}{4}({}^+\tau) \int d^4x \epsilon^{\mu\nu\alpha\beta} {}^+\mathcal{R}_{\mu\nu}^{\tau\lambda} {}^+\mathcal{R}_{\alpha\beta}^{\sigma\rho} \epsilon_{\tau\lambda\sigma\rho} - \frac{1}{4}({}^-\tau) \int d^4x \epsilon^{\mu\nu\alpha\beta} {}^-\mathcal{R}_{\mu\nu}^{\tau\lambda} {}^-\mathcal{R}_{\alpha\beta}^{\sigma\rho} \epsilon_{\tau\lambda\sigma\rho}, \quad (49)$$

where ${}^+\tau$ and ${}^-\tau$ are two different constant parameters and ${}^\pm \mathcal{F}_{\mu\nu}^{\alpha\beta}$ is given by

$${}^\pm \mathcal{F}_{\mu\nu}^{\alpha\beta} = \left(\frac{1}{2}\right)^\pm M_{\tau\lambda}^{\alpha\beta} \mathcal{F}_{\mu\nu}^{\tau\lambda}, \quad (50)$$

where

$${}^\pm M_{\tau\lambda}^{\alpha\beta} = \frac{1}{2}(\delta_{\tau\lambda}^{\alpha\beta} \mp i\epsilon^{\alpha\beta}{}_{\tau\lambda}). \quad (51)$$

It turns out that ${}^+ \mathcal{F}_{\mu\nu}^{\alpha\beta}$ is self-dual, while ${}^- \mathcal{F}_{\mu\nu}^{\alpha\beta}$ is anti self-dual curvature. Therefore, the action (49) describes self-dual and anti-self-dual linearized gravity.

6.-FINAL COMMENTS

In this article, we have investigated different aspects of linearized gravity as a gauge theory. We showed that linearized gravity can be understood as an abelian gauge theory only in connection with the subgroup $S(n-1, 1)$ but not in connection with the full group $S(n, 1)$. We apply our observations to the case of MacDowell-Mansouri theory in four dimensions showing that our proposed method leads to the Fierz-Pauli action in the weak field limit. Furthermore, we argued that our results can be directly used in the case of S-duality for linearized gravity.

In fact, in reference [1] it was discovered that the S-duality for linearized gravity leads to the cosmological constant duality $\Lambda \rightarrow \frac{1}{\Lambda}$, in a similar form as an abelian gauge theory leads to the duality of the gauge coupling constant $g^2 \rightarrow \frac{1}{g^2}$. The results of the present work seems to confirm this analogy since we showed that in a strictly sense linearized gravity can be in fact understood as an Abelian gauge theory. Moreover, it is evident from reference [1] that instead of the duality $\Lambda \rightarrow \frac{1}{\Lambda}$ it is possible to consider the duality symmetry $l_p \rightarrow \frac{1}{l_p}$ (or the more general duality transformation

$l_p \Lambda \rightarrow \frac{1}{\Lambda l_p}$). It is worth mentioning that recently, there have been much interest in the strong coupling limit of linearized gravity [4] via the duality $l_p \rightarrow \frac{1}{l_p}$. It may be interesting for further research to see if our present study is useful in this direction.

Other possible application of the results of the present work is in connection with the non-minimal coupling for spin 3/2 field studied in reference [21]. In this reference the authors find the non-minimal coupling for spin 3/2 by applying the method of ‘square root’ to the constraints of spin 2 gauge field [22]. These constraints are obtained considering the traditional method of linearized gravity presented in section 2. Now, that we have found the route to see linearized gravity as a gauge theory it seems interesting to revisit the coupling for spin 3/2 field. This possibility may shed some light on quantum linearized supergravity and since supergravity is an essential part of M-theory [23-25] one should expect eventually to gain some better understanding of such a theory. This seems a viable route to follow because it has been proved [26] that massive spin 2 coupled to gravity is deeply connected to string theory.

Finally it has been studied [27] the role of little group in linearized gravity obtaining the result that the translational subgroup of the Wigner’s little group acts as a generator of linearized gravity only when the space time has dimension four. It may be interesting for further research to investigate this intriguing result from the point of view of the present work.

Acknowledgments

I would like to thank to M. C. Marín for helpful comments.

References

- [1] J. A. Nieto, Phys.Lett.**A262**, 274 (1999); hep-th/9910049.
- [2] C. M. Hull, Nucl. Phys. **B583**, 237 (2000); hep-th/0004195.
- [3] X. Bekaert, N. Boulanger and M. Henneaux, Phys. Rev. **D67**, 044010 (2003); hep-th/0210278.
- [4] Casini, R. Montemayor, L. F. Urrutia, Phys. Rev. **D68**, 065011 (2003); hep-th/0304228; Phys. Rev. **D66**, 085018, (2002); hep-th/0206129
- [5] M. Varadarajan, Phys. Rev. **D66**, 024017 (2002); gr-qc/0204067; A. Ashtekar, C. Rovelli and L. Smolin, Phys. Rev. **D44**, 1740 (1991).
- [6] S. W. MacDowell and F. Mansouri, Phys. Rev. Lett. **38**, 739 (1977).

- [7] S. B. Giddings, E. Katz and L. Randall, JHEP **0003**, 023 (2000); hep-th/0002091; H. Collins and B. Holdom, Phys. Rev. **D62**, 124008 (2000); hep-th/0006158; N. Deruelle and T. Dolezel, Phys. Rev. **D64**, 103506 (2001); gr-qc/0105118
- [8] J.A. Nieto, J. Saucedo and V.M. Villanueva, Phys. Lett. **A312**, 175 (2003); hep-th/0303123.
- [9] T. Fukai and K. Okano, Prog. Theor. Phys., **73**, 790 (1985)
- [10] J.B. Hartle, Phys. Rev. **D29**, 2730 (1984)
- [11] A. Higuchi, Class. Quant. Grav., **8**, 2005 (1991).
- [12] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).
- [13] I.Ya. Aref'eva, M. G. Ivanov, W. Muck, K.S. Viswanathan and I.V. Volovich, Nucl. Phys. **B590**, 273 (2000): hep-th/0004114.
- [14] E. E. Boos, Y. A. Kubyshin, M. N. Smolyakov and Igor P. Volobuev, "Effective Lagrangian for linearized gravity in Randall-Sundrum model", hep-th/0105304.
- [15] I. Giannakis, J. T. Liu and H. Ren, Nucl. Phys. **B654**, 197 (2003); hep-th/0211196.
- [16] N. Deruelle, "Linearized gravity on branes from Newton's Law to cosmological perturbations", Contributed to Spanish Relativity Meeting on Gravitation and Cosmology (ERE 2002), Mao, Menorca, Spain, 22-24 Sep 2002; gr-qc/0301036
- [17] M. Novello and R.P. Neves, Class. Quant. Grav. **20**, L67 (2003); M. Novello, S. E. Perez Bergliaffa, R. P. Neves, "Replay acausality of massive charged spin-2 field"; gr-qc/0304041.
- [18] S. Deser, A. Waldron, "Acausality of massive charged spin-2 field", hep-th/0304050.
- [19] M. Fierz and W. Pauli, Helvetica Physica Acta, **12**, 297 (1939); Proc. Roy. Soc. **173A**, 211 (1939).
- [20] D. Lovelock, J. Math. Phys. **12**, 498 (1971).
- [21] V. M. Villanueva, J. A. Nieto and O. Obregon, Found. Phys. **33**, 735 (2003); Rev. Mex. Fis. **48**, 123 (2002): hep-th/0109104.
- [22] J. A. Nieto and O. Obregón, Phys. Lett. **A 175**, 11 (1993).

- [23] P. K. Townsend, “Four lectures on M-theory,” *Proceedings of the ICTP on the Summer School on High Energy Physics and Cosmology*, June 1996, hep-th/9612121.
- [24] M. J. Duff, Int. J. Mod. Phys. **A11**, 5623 (1996), hep-th/9608117.
- [25] P. Horava and E. Witten, Nucl. Phys. **B460**, 506 (1996).+
- [26] I. L. Buchbinder, D. M. Gitman, V. A. Krykhtin and V. D. Pershin, Nucl. Phys. **B584**, 615 (2000); hep-th/9910188
- [27] T. Scaria and B. Chakraborty, Class. Quant. Grav. **19**, 4445 (2002); hep-th/0205018